Universes in simplicial type theory

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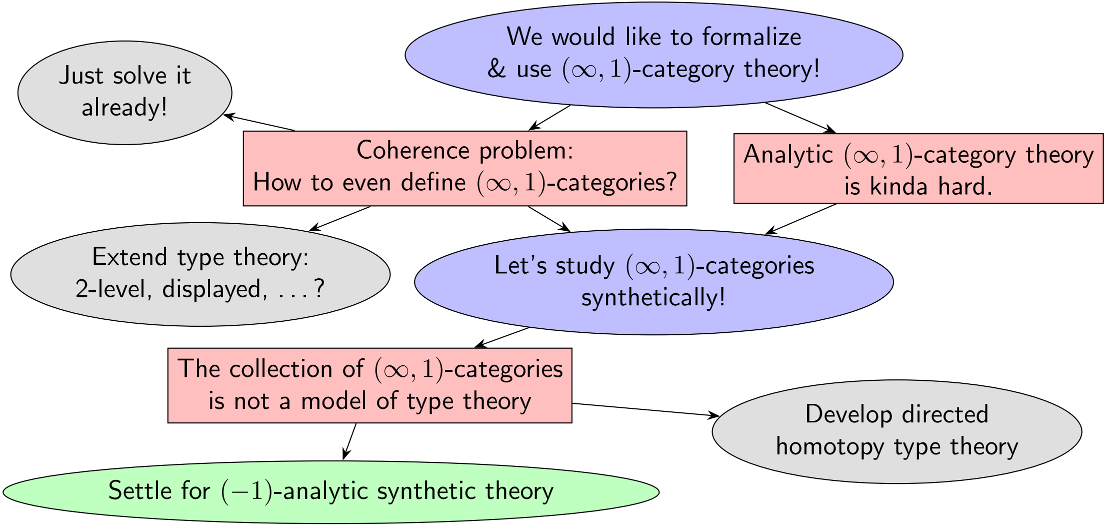
Introduction & Motivation

In the context of Homotopy Type Theory/Univalent Foundations:



Simplicial type theory

In simplicial type theory [RS17] we interpret HoTT in simplicial spaces (or simplicial objects of any higher topos). These are probed by the simplices Δ^n.

From an interval type $\mathbb{I}$ (a totally ordered set with distinct 0 and 1), we get the type of arrows $X^{\mathbb{I}}$ for any type X.

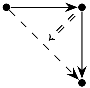
Definition X is *Segal* if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.

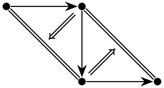
Definition A Segal type X is *Rezk* if $X^{\mathbb{E}} \to X$ is an equivalence, for $\mathbb{E}$ the "walking equivalence".

The Rezk types are our $(\infty, 1)$-categories. It's possible to do quite a lot of abstract category theory with these, incl. fibered category theory [BW23].

Proof assistant Rzk [Kud23], formalization of a version of the Yoneda lemma [KRW04].

$\Lambda^2_1 \hookrightarrow \Delta^2$





$\mathbb{E}$

Modal simplicial type theory

But there aren't many *examples* of categories in bare simplicial type theory.
In particular, we'd like the category of spaces $\mathcal{S}$.

We define this using the [LOPS18] construction, but we need to be in a setting where $\mathbb{I}$ is *tiny*, which it isn't in simplicial spaces, but it is in cubical spaces.

With MTT [GKNB21; GKNB20], we add modalities

- idempotent comonad g (global) with right adjoint s,
- involutive o (opposite),
- codistributive colattice & commutative monoid ρ (for $\mathbb{I} \to \_$),
- right adjoint $\rho \dashv \bar{\rho}$,
- τ (twisted arrows) with 2-cells $\tau \to \mathrm{id}$, $\tau \to o$,
- equations $sg = s$, $gs = g\bar{\rho} = go = g$, $\tau o = o\tau$, . . .

Enforce $\Gamma.\{\rho\} = \Gamma.\mathbb{I}$ so $\mathbb{I}$ becomes a distributive lattice with duality involution $\neg : \mathbb{I}^{\mathrm{op}} \to \mathbb{I}$.

$\mathbb{I}$ is tiny if $\mathbb{I} \to \_$ has a right adjoint.

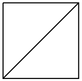
See also Riley's [Ril24]

(Cf. also [WL20])

Triangulated type theory

Cubical spaces contain simplicial spaces as a subtopos [SW21], which we can define as the lex modality $\boxslash$, nullification at the family of propositions:

$$\Phi : \mathbb{I} \times \mathbb{I} \to \mathsf{Prop}, \qquad \Phi(i,j) = (i \leq j) \vee (j \leq i)$$

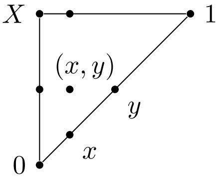


The $\boxslash$-modal types are the simplicial spaces. We add some axioms to capture that cubes detect equivalences [MR23].

Further, we use synthetic quasicoherence [Ble23], which relies on the fact that $\mathbb{I}$, qua cubical space, is simplicial, see also [Spi16].

The interval $\mathbb{I}$ is a distributive lattice with $0, 1$. Define an $\mathbb{I}$-algebra to be a distributive lattice R with $0, 1$ and a structure-preserving map $\mathbb{I} \to R$.

Axiom (Duality) For any finitely presented $\mathbb{I}$-algebra R, the map $R \to (\hom_{\mathbb{I}}(R, \mathbb{I}) \to \mathbb{I})$ is an equivalence.



Cor. (Phoa's principle) We have equivalences

$$\Delta^2 \simeq \mathbb{I}[X] \simeq (\mathbb{I} \to \mathbb{I}).$$

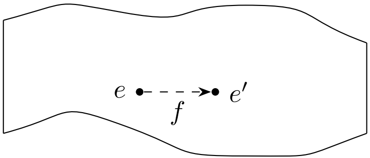
(Amazingly) Covariant Families

Consider the notion of covariant families over the interval:

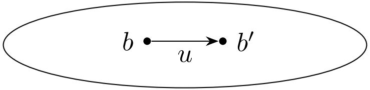
$\mathrm{Cov} : \mathcal{U}^{\mathbb{I}} \to \mathcal{U}$
$\mathrm{Cov}\ X = \prod_{x_0 : X 0} \mathrm{isContr}\big(\sum_{x_1 : X 1} \mathrm{hom}_{\mathsf{id}}(x_0, x_1)\big)$

Definition: A family $A : X \to \mathcal{U}$ is *amazingly covariant* if we have

$$\Big(\prod_{x : (i:\mathbb{I}) \to X^{\eta} \cdot i} \mathrm{Cov}(\lambda i.\, (A^{\eta} \cdot i)(x\, i))\Big)_{\mathbb{I}}$$



Theorem $\mathcal{U}_{\mathrm{Cov}} := \mathcal{U} \times_{\mathcal{U}_{\mathbb{I}}} \mathcal{U}_{\mathbb{I}}^{\bullet} = \sum_{A:\mathcal{U}} \mathrm{Cov}(\lambda i.\, A^{\eta} \cdot i)_{\mathbb{I}}$ classifies amazingly covariant families.



Definition $\mathcal{S} := \sum_{A:\mathcal{U}_{\boxslash}} \mathrm{Cov}(\lambda i.\, A^{\eta} \cdot i)_{\mathbb{I}}$.

Closure properties $\mathcal{S}$:

- contains global discrete types,
- is closed under Σ and identity types,
- is closed under finite limits and colimits.

Directed Univalent Universe of Covariant Families

To prove directed univalence we introduce directed glue types, given a type X and a family $A : X \to \mathcal{S}$:

$\mathsf{Gl}\, A : X \times \mathbb{I} \to \mathcal{U}_{\boxslash}$
$\mathsf{Gl}\, A\, x\, i = (i = 0) \to A\, x$

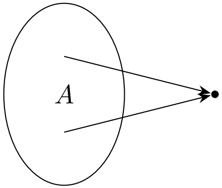
Theorem $\mathsf{Gl}\, A :_g X \times \mathbb{I} \to \mathcal{S}$.
This uses Phoa's principle.

Given $E_0, E_1 : \mathcal{S}$ with $f : E_0 \to E_1$ we get a type family

$$\textstyle\sum_{e_1 : E_1} \mathsf{Gl}\, f^{-1}(e_1) : \mathbb{I} \to \mathcal{S}$$

This gives us straightening.

Theorem (Directed univalence)
$S : (E_0 \to E_1) \to \hom_{\mathcal{S}}(E_0, E_1)$ is an equivalence.



Corollary $\mathcal{S}$ is Segal and Rezk, hence a category.

Theorem (external) $\mathcal{S}$ is simplicial.

Directed structure identity principle

As a consequence of directed univalence we can construct many other categories with the expected morphisms:

- $(\infty, 1)$-category of pointed spaces, $\sum_{X:\mathcal{S}} X$,
- 1-category of finite sets,
- 1-categories of usual algebraic structures,
- 1-category of posets, ω, Δ,
- $(\infty, 1)$-category of spectra (two definitions),
- $(\infty, 1)$-category of spaces with an endomorphism,
- $\ldots$

Conclusions and Further Work

Next steps:

- Naive Yoneda via twisted arrow modality.
- $(\infty, 1)$-category of $(\infty, 1)$-categories, functors and natural equivalences.
- Applications to higher algebra.
- Integration in Rzk.
- Computational meaning [WABN22].

Thank you

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